A NOTE ON CERTAIN SINGULAR INTEGRAL EQUATIONS WITH KERNELS $K(z, \zeta)/(z-\zeta)$

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The debate, which arose in 1965–66 between Peters (1965) and Case (1966), regarding the use of "standard" method of solving certain singular Integral Equations with Kernels $K(z, \zeta)/(z-\zeta)$, has been resolved in the present note. It has been shown that if Peter's conditions are satisfied by the given functions in the integral equation, then the singular integral equation in question can be solved in a quicker way without making any modification to the standard method.

Introduction

There has been a debate regarding the use of the "standard" method of solving certain singular integral equations (cf. Peters 1965 and Case 1966).

The "standard" method of solving the integral equation

$$\oint_C \frac{K(z, \zeta)\,\phi(z)}{z-\zeta}\,dz = h(\zeta)\,\phi(\zeta) + f(\zeta) \qquad \ldots(1)$$

involves two steps:

(i) Reduction of the integral equation to a Hilbert-Riemann boundary value problem for sectionally holomorphic functions, and

(ii) Solution of the Hilbert-Riemann boundary value problem through function-theoretic methods.

To solve the associated Hilbert-Riemann boundary value problem, one has to first solve the corresponding homogeneous equation and then the solution of the inhomogeneous equation is found out with the help of the solution of the homogeneous equation. This procedure of solving the inhomogeneous Hilbert-Riemann problem is described in the texts of Muskhelishvili (1953) and Gakhov (1958). Finally the solution of the Hilbert-Riemann problem is used to find out the solution of the integral equation (1), in the usual manner, through the Plemelj formulae (cf. Muskhelishvili 1953).

Peters (1965) has shown that if certain conditions are satisfied by the given functions, a different function-theoretic method, quick and simple, can be used to

solve singular integral equations of the type (1). In a later paper, Case (1966) has raised certain questions regarding the "standard" method and the method of Peters.

In the present note we have shown that if the conditions mentioned by Peters are satisfied by the given functions, then the associated Hilbert-Riemann boundary value problem need not be solved in the usual way. In such cases the associated Hilbert-Riemann boundary value problem can be solved quickly in a different manner. We have thus shown that the "standard" method needs no modification for quick application, if the conditions of Peters (1965) or Case (1966) are satisfied.

In Section 2, the Hilbert-Riemann boundary value problem arising from the integral equation

$$\oint_C \frac{\phi(z)}{z-\zeta}\, dz = h(\zeta)\,\phi(\zeta) + f(\zeta)$$

has been solved.

In Section 3, the case for the more general equation (1) has been dealt with.

We thus observe that the quickness of the method of solution of the singular integral equations (1) mainly depends on the types of conditions to be satisfied by the given functions.

2. The Kernel $1/(z-\zeta)$

Let us consider the equation

$$\oint_C \frac{\phi(z)}{z-\zeta}\, dz = h(\zeta)\,\phi(\zeta) + f(\zeta) \qquad \dots(2.1)$$

where C is a closed contour dividing the complex plane into an interior region D^+ and an exterior region D^-. The integration is taken such that, D^+ always lies to the left of C.

We assume that (cf. Peters 1965 and Case 1966):

(i) The functions h and f satisfy a uniform Holder condition on C;

(ii) $h(\zeta) \pm \pi i \neq 0$ for ζ on C.

Introducing the function

$$F(w) = \oint_C \frac{\phi(z)\, dz}{z-w} \qquad \dots(2.2)$$

and using the Plemelj formulae:

$$F^+(\zeta) = \pi i \phi(\zeta) + \oint_C \frac{\phi(z)\, dz}{z-\zeta} \qquad \dots(2.3)$$

$$F^{-}(\zeta) = -\pi i \phi(\zeta) + \oint_C \frac{\phi(z)\, dz}{z-\zeta}, \qquad \text{...(2.4)}$$

eqn. (2.1) can be reduced to the Hilbert-Riemann boundary value problem of finding $F(w)$ such that the equation

$$[h(\zeta) - \pi i] F^{+}(\zeta) - [h(\zeta) + \pi i] F^{-}(\zeta) = -2\pi i f(\zeta) \qquad \text{...(2.5)}$$

is satisfied.

To solve this associated problem (2.5) we proceed in a manner given below which is different from the usual procedure.

Since $F^{-}(\zeta) = F^{+}(\zeta) - 2\pi i \phi(\zeta)$, eqn. (2.5) can be written as

$$\phi(\zeta) = \frac{F^{+}(\zeta) - f(\zeta)}{h(\zeta) + \pi i}. \qquad \text{...(2.6)}$$

Substituting this value of ϕ in (2.2) we obtain:

$$F(w) = \oint_C \frac{F^{+}(z)\, dz}{[h(z) + \pi i](z-w)} - \oint_C \frac{f(z)\, dz}{[h(z) + \pi i](z-w)}. \qquad \text{...(2.7)}$$

Let us now assume, as in Peters (1965), that $h(z) + \pi i$ is analytic in $D^{+} + C$ with simple zeros in D^{+} at $\alpha_K, K = 1, 2, \ldots n_1$.

Then

$$F(w) = 2\pi i \frac{F^{-}(w)}{h(w) + \pi i} + \sum_{K=1}^{n_1} \frac{C_K}{\alpha_K - w} - \oint_C \frac{f(z)\, dz}{[h(z) + \pi i](z-w)}$$

where C_k is $2\pi i$ times the residue of $\dfrac{f(z)}{h(z) + \pi i}$ at $z = \alpha_k$.

Thus

$$F(w) = \frac{h(w) + \pi i}{h(w) - \pi i} \left[\sum_{K=1}^{n_1} \frac{C_K}{\alpha_K - w} - \oint_C \frac{f(z)\, dz}{[h(z) + \pi i](z-w)} \right]. \qquad \text{...(2.8)}$$

By Plemelj formula, we obtain, from (2.8)

$$F^{+}(\zeta) = \frac{h(\zeta) + \pi i}{h(\zeta) - \pi i} \left[\sum_{K=1}^{n_1} \frac{C_K}{\alpha_K - \zeta} - \frac{\pi i f(\zeta)}{h(\zeta) + \pi i} - \oint_C \frac{f(z)\, dz}{[h(z) + \pi i](z-\zeta)} \right]. \qquad \text{...(2.9)}$$

Substituting this value of $F^+(\zeta)$ from (2.9) into (2.6) we get

$$\phi(\zeta) = -\frac{h(\zeta) f(\zeta)}{h^2(\zeta) + \pi^2} - \frac{1}{h(\zeta) - \pi i} \oint_C \frac{f(z)\,dz}{[h(z) + \pi i](z - \zeta)}$$

$$+ \frac{1}{h(\zeta) - \pi i} \sum_{K=1}^{n_1} \frac{C_K}{\alpha_K - \zeta} \qquad \ldots(2.10)$$

which agrees with eqn. (2.23) of Peters (1965).

Thus we observe that under the assumptions made by Peters, the method of solving the associated Hilbert-Riemann boundary value problem is very quick.

In the next section, we solve the associated Hilbert-Riemann boundary value problem for a more general equation involving the kernel $\dfrac{K(z, \zeta)}{z - \zeta}$.

3. The Kernel $K(z, \zeta)/(z - \zeta)$

We consider the equation

$$\oint_C \frac{K(z, \zeta)\ \phi(z)}{z - \zeta}\,dz = h(\zeta)\,\phi(\zeta) + f(\zeta) \qquad \ldots(3.1)$$

where (Case 1965)

(i) C, $\phi(\zeta)$ and $f(\zeta)$ are required to satisfy the conditions for eqn. (2.1)

(ii) $h(\zeta) \pm \pi i K(\zeta, \zeta) \neq 0$ for ζ on C.

(iii) $K(\zeta, \zeta) \neq 0$ for ζ on C.

(iv) $K(z, w)$ is analytic for both z and w in $D^+ + C$.

Writing

$$K(z, \zeta) = [K(z, \zeta) - K(z, z)] + K(z, z)$$

eqn. (3.1) can be written as

$$\oint_C \frac{\psi(z)\,dz}{z - \zeta} = \frac{h(\zeta)}{K(\zeta, \zeta)}\,\psi(\zeta) + H(\zeta) \qquad \ldots(3.2)$$

where

$$\psi(z) = K(z, z)\,\phi(z)$$

and

$$H(\zeta) = f(\zeta) + \oint_C \frac{K(z,\ z) - K(z,\ \zeta)}{z - \zeta}\,\phi(z)\,dz = f(\zeta) + g(\zeta), \quad \text{(say)}$$

where

$$g(\zeta) = \oint_C \frac{K(z, z) - K(z, \zeta)}{z - \zeta} \phi(z)\, dz.$$

Assuming $H(\zeta)$ known and introducing

$$F(w) = \oint_C \frac{\psi(z)\, dz}{z - w} \qquad \text{...(3.3)}$$

we get, using Plemelj formula and (3.2),

$$\psi(\zeta) = \frac{K(\zeta, \zeta)\, [F^+(\zeta) - H(\zeta)]}{h(\zeta) + \pi i K(\zeta, \zeta)}. \qquad \text{...(3.4)}$$

Substituting this value of ψ in (3.3) we obtain:

$$F(w) = \oint_C \frac{K(z, z)\, F^+(z)\, dz}{[h(z) + \pi i K(z, z)]\,(z - w)}$$

$$- \oint_C \frac{K(z, z)\, H(z)\, dz}{[h(z) + \pi i K(z, z)]\,(z - w)}. \qquad \text{...(3.5)}$$

If, now, we assume that $h(z) + \pi i K(z, z)$ has simple zeros in D^+ situated at $\alpha_K, K = 1, 2, \ldots n_1$ then

$$F(w) = \frac{2\pi i K(w, w)\, F^+(w)}{h(w) + \pi i K(w, w)} + \sum_{K=1}^{n_1} \frac{C_K}{\alpha_K - w}$$

$$- \oint_C \frac{K(z, z)\, H(z)}{[h(z) + \pi i K(z, z)]} \frac{dz}{(z - w)},$$

where C_K, $K = 1, 2, \ldots n_1$ are the residues of $\dfrac{K(z, z)\, F^+(z)}{h(z) + \pi i K(z, z)}$ at α_K. Thus

$$F(w) = \frac{h(w) + \pi i K(w, w)}{h(w) - \pi i K(w, w)} \left[\sum_{K=1}^{n_1} \frac{C_K}{\alpha_K - w} \right.$$

$$\left. - \oint_C \frac{K(z, z)\, H(z)\, dz}{[h(z) + \pi i K(z, z)]\,(z - w)} \right]. \qquad \text{...(3.6)}$$

Applying Plemelj formula to (3.6) we obtain:

$$F^+(\zeta) = \frac{h(\zeta) + \pi i K(\zeta, \zeta)}{h(\zeta) - \pi i K(\zeta, \zeta)} \left[\sum_{K=1}^{n_1} \frac{C_K}{\alpha_K - \zeta} - \frac{\pi i K(\zeta, \zeta)\, H(\zeta)}{h(\zeta) + \pi i K(\zeta, \zeta)} \right.$$

$$\left. - \oint_C \frac{K(z, z)\, H(z)\, dz}{[h(z) + \pi i K(z, z)]\,(z - \zeta)} \right]. \qquad \text{...(3.7)}$$

Substituting this value of $F^{+}(\zeta)$ in (3.4) we get:

$$\phi(\zeta) = -\frac{h(\zeta)\,H(\zeta)}{h^2(\zeta) + \pi^2 K^2(\zeta,\zeta)} - \frac{1}{h(\zeta) - \pi i K(\zeta,\zeta)}$$

$$\oint_C \frac{K(z,z)\,H(z)\,dz}{[h(z) + \pi i K(z,z)]\,(z-\zeta)} + \frac{1}{h(\zeta) - \pi i K(\zeta,\zeta)} \sum_{K=1}^{n_1} \frac{C_K}{\alpha_K - \zeta}$$

$$= -\frac{h(\zeta)\,H(\zeta)}{h^2(\zeta) + \pi^2 K^2(\zeta,\zeta)} - \frac{1}{h(\zeta) - \pi i K(\zeta,\zeta)} \oint_C \frac{K(\zeta z, z)\,f(z)\,dz}{[h(z) + \pi i K(z,z)]\,(z-\zeta)}$$

$$- \frac{\pi i K(\zeta,\zeta)\,g(\zeta)}{h^2(\zeta) + \pi^2 K^2(\zeta,\zeta)} - \frac{1}{h(\zeta) - \pi i K(\zeta,\zeta)} \sum_{K=1}^{n_1} \frac{C_K'}{\alpha_K - \zeta}$$

$$+ \frac{1}{h(\zeta) - \pi i K(\zeta,\zeta)} \cdot \sum_{K=1}^{n_1} \frac{C_K}{\alpha_K - \zeta},$$

where C_K''s are the residues of $\dfrac{K(z,z)\,g(z)}{h(z) + \pi i K(z,z)}$ at α_K.

We thus arrive at

$$\phi(\zeta) = -\frac{h(\zeta)\,f(\zeta)}{h^2(\zeta) + \pi^2 K^2(\zeta,\zeta)} - \frac{g(\zeta)}{h(\zeta) - \pi i K(\zeta,\zeta)}$$

$$- \frac{1}{h(\zeta) - \pi i K(\zeta,\zeta)} \oint_C \frac{K(z,z)\,f(z)}{[h(z) + \pi i K(z,z)]\,(z-\zeta)}$$

$$- \left(\sum_{K=1}^{n_1} \frac{d_K}{\alpha_K - \zeta} \right) \frac{1}{h(\zeta) - \pi i K(\zeta,\zeta)} \cdot (d_K = C_K' - C_K). \quad \ldots(3.8)$$

Multiplying both sides of (3.8) by $\dfrac{\{K(\zeta,\zeta) - K(\zeta,\zeta')\}}{(\zeta - \zeta')}$ and integrating with respect to ζ over C we obtain:

$$g(\zeta') = -\oint_C \frac{h(\zeta)\,f(\zeta)}{h^2(\zeta) + \pi^2 K^2(\zeta,\zeta)} \cdot \frac{\{K(\zeta,\zeta) - K(\zeta,\zeta')\}}{(\zeta - \zeta')}\,d\zeta$$

$$- \oint_C \frac{g(\zeta)}{[h(\zeta) - \pi i K(\zeta,\zeta)]} \cdot \frac{\{K(\zeta,\zeta) - K(\zeta,\zeta')\}}{(\zeta - \zeta')}\,d\zeta$$

$$- \oint_C \frac{\{K(\zeta,\zeta) - K(\zeta,\zeta')\}\,d\zeta}{[h(\zeta) - \pi i K(\zeta,\zeta)]\,(\zeta - \zeta')} \oint_C \frac{K(z,z)\,f(z)\,dz}{[h(z) + \pi i K(z,z)]\,(z-\zeta)}$$

$$- \oint_C \frac{\{K(\zeta,\zeta) - K(\zeta,\zeta')\}}{[h(\zeta) - \pi i K(\zeta,\zeta)]\,(\zeta - \zeta')} \cdot \left(\sum_{K=1}^{n_1} \frac{d_K}{\alpha_K - \zeta}\right) d\zeta$$

$$= - \oint_C \frac{h(\zeta) f(\zeta)\,[K(\zeta,\zeta) - K(\zeta,\zeta')]\,d\zeta}{[h^2(\zeta) + \pi^2 K^2(\zeta,\zeta)]\,(\zeta - \zeta')} - \sum_{K=1}^{n_2} \frac{a_K}{\beta_K - \zeta'}$$

$$- \oint_C \frac{K(\zeta z, z) f(z)}{\{h(z) + \pi i K(z,z)\}} \oint_C \frac{\{K(\zeta,\zeta) - K(\zeta,\zeta')\}}{[h(\zeta) - \pi i K(\zeta,\zeta)]\,(\zeta - \zeta')(z - \zeta)}\, d\zeta dz$$

$$- \sum_{K=1}^{n_1} d_K \oint_C \frac{\{K(\zeta,\zeta) - K(\zeta,\zeta')\}\, d\zeta}{[h(\zeta) - \pi i K(\zeta,\zeta)]\,(\zeta - \zeta')(\alpha_K - \zeta)},$$

where β_K, $K = 1, 2, \ldots n_2$ are the simple zeros of $h(z) - \pi i K(z,z)$ inside C and $a_K, K = 1, 2, \ldots n_2$ are the residues of $\dfrac{g(z)\,[K(z,z) - K(z,\zeta)]}{h(z) - \pi i K(z,z)}$, at β_K.

We finally arrive at:

$$g(\zeta') = - \oint_C \frac{h(\zeta) f(\zeta)}{h^2(\zeta) + \pi^2 K^2(\zeta,\zeta)} \cdot \frac{\{K(\zeta,\zeta) - K(\zeta,\zeta')\}}{(\zeta - \zeta')}\, d\zeta - \sum_{K=1}^{n_2} \frac{a_K}{\beta_K - \zeta'}$$

$$+ \pi i \oint_C \frac{K(z,z) f(z)}{h^2(z) + \pi^2 K^2(z,z)} \cdot \frac{\{K(z,z) - K(z,\zeta')\}}{(z - \zeta')}\, dz$$

$$- \oint_C \frac{K(z,z) f(z)}{h(z) + \pi i K(z,z)} \left(\sum_{K=1}^{n_2} \frac{b_k}{(\beta_K - \zeta')(z - \beta_K)}\right) dz$$

$$+ \sum_{K=1}^{n_1} d_K \cdot \frac{A}{(\alpha_K - \zeta')} - \sum_{K=1}^{n_1} d_K \sum_{K'=1}^{n_2} \frac{b_{K'}}{(\beta_{K'} - \zeta')(\alpha_K - \beta_{K'})},$$

(A = a constant),

where b_i, $i = 1, 2, \ldots n_2$ are the residues of $\dfrac{\{K(\zeta,\zeta) - K(\zeta,\zeta')\}}{h(\zeta) - \pi i K(\zeta,\zeta)}$ at β_i.

Thus

$$g(\zeta') = - \oint_C \frac{f(\zeta)}{h(\zeta) + \pi i K(\zeta,\zeta)} \cdot \frac{\{K(\zeta,\zeta) - K(\zeta,\zeta')\}}{(\zeta - \zeta')}\, d\zeta$$

$$- \oint_C \frac{K(z,z) f(z)\, dz}{h(z) + \pi i K(z,z)} \left(\sum_{K=1}^{n_2} \frac{b_K}{(\beta_K - \zeta')(z - \beta_K)}\right) - \sum_{K=1}^{n_2} \frac{a_K}{\beta_K - \zeta'}$$

$$+ \sum_{K=1}^{n_1} \frac{A d_K}{(\alpha_K - \zeta')} - \sum_{K=1}^{n_2} d_K \sum_{K'=1}^{n_1} \frac{b_{K'}}{(\beta_{K'} - \zeta')(\alpha_K - \beta_{K'})}. \qquad \ldots(3.9)$$

Inserting this value of $g\,(')$ into (3.8) we obtain:

$$\phi(\zeta) = -\frac{h(\zeta)f(\zeta)}{h^2(\zeta)+\pi^2K^2(\zeta,\zeta)} - \frac{1}{\{h(\zeta)-\pi iK(\zeta,\zeta)\}}$$

$$\times \oint_C \frac{K(z,\zeta)f(z)\,dz}{[h(z)+\pi iK(z,z)]},\ \frac{1}{(z-\zeta)} + \frac{1}{h(\zeta)-\pi iK(\zeta,\zeta)}\,.$$

$$\times \oint_C \frac{K(z,z)f(z)\,dz}{h(z)+\pi iK(z,z)}\left(\sum_{K=1}^{n_2}\frac{b_K}{(\beta_K-\zeta)(z-\beta_K)}\right)$$

$$-\frac{1}{h(\zeta)-\pi iK(\zeta,\zeta)}\cdot\left[\sum_{K=1}^{n_2}\frac{a_k}{\beta_k-\zeta}+\sum_{K=1}^{n_1}\frac{Bd_K}{\alpha_K-\zeta}\right.$$

$$\left.-\sum_{K=1}^{n_1}d_k\sum_{K'=1}^{n_2}\frac{b_{K'}}{(\alpha_K-\beta_{K'})(\beta_{K'}-\zeta)}\right]\quad (B=\text{a constant}). \qquad \ldots(3.10)$$

The result (3.10) agrees with the corresponding result of Peters (1965).

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